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Multiplicative stochastic differential equations with noise-induced transitions

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Abstract. We investigate a class of linear multiplicative stochastic differential equations and demonstrate the existence of a striking noise-induced transition in the structure of the resulting asymptotic stationary probability distribution for the dependent variable. The transition amounts to a change from a bounded distribution to an unbounded one with only a finite number of convergent moments. It occurs when the range of fluctuation of one of the variables driven by the underlying stochastic process increases sufficiently to permit changes of sign for the variable. It seems likely that the phenomenon is a general one and occurs in a wider class of models than that discussed in this paper. We obtain explicit results for simple cases which we confirm by appropriate numerical simulations. This gives us the opportunity of assessing the applicability of perturbation theory which is one of the few calculational methods employed on these models up until now.

1. Introduction

There are many applications of stochastic differential equations [1, 2]. One motivation for our investigation is our interest in the Pope equation [3, 4] for the development of curvature of material elements swept along in random and turbulent flows [5, 6]. A second application with which we are concerned is to the 'slave equations' which are part of the Langevin simulation method for evaluating quantum field theory and statistical field theoretic systems [7]. Both these applications involve equations which can be interpreted as linear multiplicative stochastic equations. These motivating applications are not important for the purposes of this paper; however we feel that the subject and the results are of more general interest than these particular applications and therefore worth presenting separately. In this paper then we shall be concerned with linear multiplicative equations. In principle the ideas could with appropriate modifications be applied to any polynomial type of equation.

In section 2 we introduce the model stochastic differential equations we wish to study. Detailed analysis of these equations is presented in sections 3 and 4. We apply the methods to some simple explicit examples in section 5 and illustrate these results with a few numerical simulations in section 6. In section 7 we examine limits in which the correlation time of the underlying stochastic process becomes either very small or very large relative to other time scales in the problem. We assess the applicability of perturbation methods based on those limits. We conclude with a brief discussion of our results together with some suggestions for extending and generalizing the analysis.

2. Linear multiplicative stochastic differential equations

We wish to study equations of the form

$$\dot{y} = -A(t)y + B(t). \tag{1}$$

In its most general form y and $B(t)$ would be vectors and $A(t)$ a matrix. We comment on this general version of the problem at the end of the paper but for the most part we wish to consider the one-dimensional case where y, A and B are scalar quantities. The idea behind the equation is that $A(t)$ and $B(t)$ are subject to statistical fluctations. From a knowledge of the statistical properties of A and B we are to infer the statistical properties of y implied by equation (1). We will investigate the following model for the ensemble governing the statistical properties of A and B. We assume they are both functions of a Markovian stochastic process $\xi(t)$ that takes values in a parameter space which may be discrete. The process ξ is characterized by a stationary probability distribution $Q(\xi)$ and develops in time by changing its value with a constant probability per unit time of τ^{-1}. This means that ξ holds its value constant during each of a sequence of time intervals the lengths of which are independently distributed according to a Poisson distribution. Within each interval ξ is distributed independently according to $Q(\xi)$.

For any stochastic process we can define an operator Δ, which plays the role of the Laplacian in standard diffusion processes, by the requirement

$$\Delta f(\xi) = \lim_{h\to 0} E[f(\xi(t+h)) - f(\xi(t))|\xi(t) = \xi] \tag{2}$$

where $E[X|Y]$ means the expectation of X subject to condition Y. The equation for the time development of the probability distribution $P(\xi, t)$ is

$$\frac{\partial}{\partial t} P(\xi, t) = \Delta^{\dagger} P(\xi, t) \tag{3}$$

where $\Delta^{\dagger}$ is the Hermitian conjugate operator to Δ. For the process $\xi(t)$ it is easy to verify that

$$\Delta f(\xi) = \frac{1}{\tau}\left(\int \mathrm{d}\xi' Q(\xi') f(\xi') - f(\xi)\right) \tag{4}$$

where $\mathrm{d}\xi$ is the appropriate measure over the parameter space for ξ. The Hermitian conjugate operator is given by

$$\Delta^{\dagger} f(\xi) = \frac{1}{\tau}\left(Q(\xi)\int \mathrm{d}\xi' f(\xi') - f(\xi)\right). \tag{5}$$

Equation (3) becomes

$$\frac{\partial}{\partial t} P(\xi, t) = \frac{1}{\tau}\left(Q(\xi)\int \mathrm{d}\xi' P(\xi', t) - P(\xi, t)\right). \tag{6}$$

Clearly $P(\xi, t) = Q(\xi)$ is indeed the static solution of this equation. Equation (6) is often referred to as a master equation for $P(\xi, t)$.

Expectation values in the static limit of a quantity $F(\xi(t))$ is

$$\langle F(\xi(t))\rangle = \int \mathrm{d}\xi Q(\xi) F(\xi). \tag{7}$$

The corresponding result for the correlation function $G(h) = \langle H(\xi(t+h))F(\xi(t))\rangle$ is

$$G(h) = \mathrm{e}^{-h/\tau}\langle H(\xi(t)F(\xi(t))\rangle + (1-\mathrm{e}^{-h/\tau})\langle H(\xi(t))\rangle\langle F(\xi(t))\rangle. \tag{8}$$

We have then

$$G_{\mathrm{C}}(h) = \mathrm{e}^{-h/\tau} G_{\mathrm{C}}(0) \tag{9}$$

where the suffix C indicates that we are dealing with the cumulant part of the correlation function. It is clear then that τ is the correlation time for the process.

When we adjoin equation (1) to the stochastic process $\xi(t)$ we have to deal with a joint probability distribution $P(\xi, y, t)$ for both ξ and y. It satisfies the 'diffusion' equation

$$\frac{\partial}{\partial t} P(\xi, y, t) = \frac{\partial}{\partial y}\left[A(\xi)y - B(\xi)\right] P(\xi, y, t) + \Delta^{\dagger} P(\xi, y, t). \tag{10}$$

The first term on the right of equation (10) is a 'drift' term obtained from the 'velocity field' for y implied by equation (1). A static solution (assuming it exists) of equation (10) satisfies

$$\frac{\partial}{\partial y}\left(A(\xi)y - B(\xi)\right) P(\xi, y) = \frac{1}{\tau}\left(P(\xi, y) - Q(\xi)\int \mathrm{d}\xi' P(\xi', y)\right) \tag{11}$$

where we have used (5) and dropped explicit mention of t.

We simplify the problem yet further by assuming that the stochastic process $\xi(t)$ assumes values in a finite set $\{\xi_1, \xi_2, \ldots, \xi_N\}$ with probabilities $\{q_1, q_2, \ldots, q_N\}$. The joint probability distribution $P(\xi, y)$ is replaced by the discrete set of probabilities $\{p_1(y), p_2(y), \ldots, p_N(y)\}$, which satisfy a discrete version of (11)

$$\frac{\partial}{\partial y}\left(\alpha_n y - \beta_n\right) p_n(y) = \frac{1}{\tau}\left(p_n(y) - q_n \sum_{m=0}^{N} p_m(y)\right) \tag{12}$$

where $\alpha_n = A(\xi_n)$ and $\beta_n = B(\xi_n)$. If we introduce an obvious matrix notation in which A and B become diagonal $N \times N$ matrices with entries $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ and $\{\beta_1, \beta_2, \ldots, \beta_N\}$ respectively and q and $p(y)$ denote column vectors of the appropriate probabilities, then (12) becomes

$$\frac{\partial}{\partial y}\left(Ay - B\right) p(y) = \frac{1}{\tau}(\mathbf{1} - qu^{\mathrm{T}})p(y) \tag{13}$$

where u is a column vector with N unit entries, u^{T} is the corresponding row vector and $\mathbf{1}$ represents the appropriate unit matrix. This is the form in which we will study our original linear multiplicative stochastic differential equation (1). Note these results

$$u^{\mathrm{T}}q = 1 \tag{14}$$

and

$$u^{\mathrm{T}}(\mathbf{1} - qu^{\mathrm{T}}) = 0 = (\mathbf{1} - qu^{\mathrm{T}}). \tag{15}$$

We also have the normalization condition

$$\int \mathrm{d}y p(y) = q. \tag{16}$$

In this matrix notation the expectation values of $A(t)$ and $B(t)$ are given by $\langle A\rangle = u^{\mathrm{T}}Aq$ and $\langle B\rangle = u^{\mathrm{T}}Bq$.

3. Analysis of the model equations

Some qualitative observations on the nature of the static probability distribution are clear from (1). If the ranges of values assumed by $A(\xi)$ and $B(\xi)$ are both finite and in addition $A(\xi)$ is strictly positive then for sufficiently large and positive y the velocity $\dot{y}$ will be negative and for sufficiently large and negative y the velocity will be positive. It follows that any y which starts out in these regions will be swept into a central region in which $\dot{y}$ acquires fluctuating values. Clearly the static probability distribution for y will be confined to this central region and vanish outside it. However when $A(\xi)$ undergoes changes of sign this argument can no longer be used. As we will see the probability distribution for the dependent variable y changes character. Its support spreads out and ceases to be confined to a central range. Applied to our simple N-component model this argument implies that when all $\{\alpha_n\}$ are strictly positive the static probability distribution has its support in the range $\min\{\beta_n/\alpha_n\} < y < \max\{\beta_n/\alpha_n\}$ but when any of $\{\alpha_n\}$ becomes negative this is no longer the case. This change in support of the probability distribution from one case to another is accompanied by changes in the character of the distribution. As will be shown later the distribution acquires an inverse power-law tail for large y. This affects the structure of the moments of the distribution. For n sufficiently large $\langle y^n\rangle$ will be divergent. This change of behaviour is of the greatest significance for the applications in which we are interested.

From (13) we see immediately that

$$\frac{\partial}{\partial y}u^{\mathrm{T}}\,(Ay - B)\,p(y) = 0. \tag{17}$$

It follows that

$$u^{\mathrm{T}}\,(Ay - B)\,p(y) = K \tag{18}$$

where K is a constant. In fact K must vanish since in order for each $p_n(y)$ to be integrable at infinity it must satisfy $yp_n(y) \to 0$ as $y \to \infty$. For polynomial equations of a degree higher than one, however, this argument no longer applies.

The analysis of (13) is clearest if we move to Fourier transform space. We define

$$x(\omega) = \int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}\omega y} p(y) \tag{19}$$

so that

$$p(y) = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega y} x(\omega). \tag{20}$$

Note that because of (16)

$$x(0) = q. \tag{21}$$

Equation (13) becomes

$$\mathrm{i}\omega \left(-\frac{\partial}{\partial \mathrm{i}\omega} A - B \right) x(\omega) = \frac{1}{\tau} \left(\mathbf{1} - q u^{\mathrm{T}} \right) x(\omega). \tag{22}$$

Because equation (18) holds with $K = 0$ there are no hidden traps in this equation and we can conclude the apparently obvious result

$$\left(-\frac{\partial}{\partial \mathrm{i}\omega} A - B \right) x(\omega) = \frac{1}{\mathrm{i}\omega\tau} \left(\mathbf{1} - q u^{\mathrm{T}} \right) x(\omega). \tag{23}$$

The point here is that were K not to vanish we would have had an extra term proportional to $K\delta(\omega)$ on the right-hand side of this equation. As it stands we see that either as (22) or (23) the equation is (in the vector sense) a standard first-order differential equation with a regular singular point at $\omega = 0$ and an irregular singular point at $\omega = \infty$. There are therefore N linearly independent solutions. We can classify the members of a basis either by their behaviour as $\omega \to 0$ or alternatively as $\omega \to \infty$. Both versions are useful.

The general theory of differential equations tells us that there is a solution of the form

$$x(\omega) = \sum_{n=0}^{\infty} (\mathrm{i}\omega)^{n+\sigma} x^{(n)} \tag{24}$$

where the coefficients $x^{(n)}$ are N-component vectors. From (22) we find for the indicial equation

$$\left(\sigma A + \frac{1}{\tau} (\mathbf{1} - q u^{\mathrm{T}}) \right) x^{(0)} = 0. \tag{25}$$

A predictable solution has $\sigma = 0$ and $x^{(0)} = q$. In general σ must satisfy

$$\det((\mathbf{1} + \tau\sigma A) - q u^{\mathrm{T}}) = \prod_{n=1}^{N} (1 + \tau\sigma\alpha_n) \left(1 - \sum_{n=0}^{N} \frac{q_n}{1 + \tau\sigma\alpha_n} \right) = 0. \tag{26}$$

Solutions for σ lie at the zeros of the factor $Y(\sigma)$ given by

$$Y(\sigma) = \left(1 - \sum_{n=0}^{N} \frac{q_n}{1 + \tau\sigma\alpha_n}\right). \tag{27}$$

For clarity of exposition we will ignore marginal situations in which one or more of the $\{\alpha_n\}$ vanish. We can recover these situations by an appropriate limiting procedure. There are then essentially two situations of interest: (i) all $\{\alpha_n\}$ positive; (ii) some $\{\alpha_n\}$ negative. If all α_n are strictly positive and distinct then all N poles of $Y(\sigma)$ lie on the negative σ-axis. Because $Y(\sigma) \to 1$ as $\sigma \to \infty$ and because the poles have negative residues there is no zero to the left of the leftmost pole. The rightmost zero is the special case $\sigma = 0$ mentioned earlier. The positions of the other zeros interlace the positions of the poles and are therefore strictly negative. When one of the $\{\alpha_n\}$ becomes negative the associated pole moves to the positive axis. Because its residue has also changed sign and become positive there is now no zero to the right of this pole. The associated zero lies to the left of the pole and will initially lie on the positive axis. As the value of the relevant α_n becomes more negative the pole moves left with the associated zero remaining to the left of the pole. The position of this zero may coincide with the special solution $\sigma = 0$ and may ultimately become negative. The condition for this coincidence is $Y'(0) = 0$. The condition for the zero to remain positive is $Y'(0) > 0$. As other α_n become negative so their associated poles move to the positive axis and their residues undergo a change of sign. The resulting situation is one in which the positive axis poles have interlacing zeros, none to the right of the rightmost pole and a remaining zero between the rightmost negative pole and the leftmost positive pole in addition to the special zero at $\sigma = 0$. To preclude this last zero from returning to the negative axis we will constrain the parameters in $Y(\sigma)$ so that $Y'(0) > 0$. The situation when two $\{\alpha_n\}$ coincide is special. In that case $Y(\sigma)$ loses one of its poles and one of its zeros. However the other factors in the expression for the determinant in (26) yield a double zero which cancels the coincident pole and turns it into the remaining zero. Essentially then there are always N solutions for σ. We will denote these solutions by $\{\sigma_1, \sigma_2, \ldots, \sigma_N,\}$ with $\sigma_1 = 0$. They may be substituted back into the indicial equation (25) to yield the corresponding solution for $x^{(0)}$. The rest of the series can be computed from the recurrence relation implied by (22), which can be put in the form

$$\left[(\mathbf{1} + \tau(n + \sigma + 1)A) - qu^{\mathrm{T}}\right] x^{(n+1)} = -\tau B x^{(n)}. \tag{28}$$

This can be solved for $x^{(n+1)}$ in terms of $x^{(n)}$ to give

$$x^{(n+1)} = D_n^{-1}\left[D_n + \frac{1}{Y(n+\sigma+1)} qu^{\mathrm{T}}\right] D_n^{-1}\,(-\tau B)\,x^{(n)} \tag{29}$$

where $D_n = \mathbf{1} + \tau(n + \sigma + 1)A$.

The asymptotic behaviour of the solutions in the limit $\omega \to \infty$ (we assume positive real ∞ for definiteness) can be analysed by substituting an appropriate asymptotic series into (22). We set

$$x(\omega) = \mathrm{e}^{-\mathrm{i}\omega s} \sum_{n=0}^{\infty} X^{(n)} (\mathrm{i}\omega)^{-n-\lambda}. \tag{30}$$

The implication of (22) for the first two terms in the series is

$$(sA - B)\, X^{(0)} = 0 \tag{31}$$

and

$$\left(\lambda A - \frac{1}{\tau}(1 - qu^{\mathrm{T}})\right) X^{(0)} + (sA - B)\, X^{(1)} = 0. \tag{32}$$

Because A and B are diagonal (31) is easily solved. The exponent s must take one of the values $\{s_n = \beta_n/\alpha_n\}$ and $X^{(0)}$ correspondingly becomes one of the set $\{e_n\}$ where e_n is a column vector with unity in the nth place and zeros everywhere else. By taking a scalar product of (32) with $X^{(0)}$ we see that it implies

$$X^{(0)\mathrm{T}} \left(\lambda A - \frac{1}{\tau}(1 - qu^{\mathrm{T}})\right) X^{(0)} = 0. \tag{33}$$

When $s = s_n$ and $X^{(0)} = e_n$ this implies that $\lambda = \lambda_n = (1 - q_n)/\tau\alpha_n$.

One way of thinking about the N linearly independent solutions of (22) is to regard them as the columns of an $N \times N$ matrix. For example we can amalgamate the N solutions defined by their behaviour near the origin into a matrix $\Phi(\omega)$ which satisfies the equation

$$\mathrm{i}\omega \left(-\frac{\partial}{\partial \mathrm{i}\omega} A - B\right) \Phi(\omega) = \frac{1}{\tau}\left(1 - qu^{\mathrm{T}}\right) \Phi(\omega). \tag{34}$$

In a similar way we could form the N solutions that are specified by their asymptotic behaviour as $\omega \sim \infty$ into a matrix $\Psi(\omega)$ which satisfies the same equation. We immediately obtain a differential equation for $\det \Phi(\omega)$ or $\det \Psi(\omega)$. It is

$$\frac{\partial}{\partial \mathrm{i}\omega} \det \Phi(\omega) = -\operatorname{Tr}\left(A^{-1}B + \frac{1}{\mathrm{i}\tau\omega} A^{-1}(1 - qu^{\mathrm{T}})\right) \det \Phi(\omega) \tag{35}$$

and $\det \Psi(\omega)$ satisfies the same equation. By normalizing the basis solutions in an appropriate way we can arrange for the solution of this equation to be

$$\det \Phi(\omega) = \det \Psi(\omega) = \exp\{-\sum_{n=1}^{N} \mathrm{i}\omega s_n\}(\mathrm{i}\omega)^{-\Lambda} \tag{36}$$

where $\Lambda = \sum_{n=1}^{N} \lambda_n$. Because both sets of solutions are complete there is a linear transformation between them represented by a matrix S such that

$$\Phi(\omega) = \Psi(\omega) S \tag{37}$$

and with these choices of normalization we see that $\det S = 1$. Now from the behaviour of the columns of $\Phi(\omega)$ near zero it can be inferred that

$$\det \Phi(\omega) \sim (\mathrm{i}\omega)^{\Sigma} \tag{38}$$

where $\Sigma = \sum_{n=1}^{N} \sigma_n$. Comparing this result with (36) as $\omega \to 0$ we can conclude that

$$\sum_{n=1}^{N} (-\sigma_n) = \sum_{n=1}^{N} \lambda_n \ . \tag{39}$$

4. Reconstruction of the probability distribution

In order to reconstruct the probability distribution $p(y)$ we make use of (22). The nature of the resulting distribution reflects the structure of $x(\omega)$ in both the limits discussed in the previous section. Consider first the case for which all the $\{\alpha_n\}$ are strictly positive. All the $\{\sigma_n\}$ are strictly negative except for σ_1 which vanishes. The corresponding solution is the only candidate that has the right properties at $\omega = 0$ to be identified with the Fourier transform of $p(y)$. As can be seen from (37) this solution is a superposition of those solutions specified by their asymptotic behaviour at large ω. That is, for an appropriate set of coefficients $\{c_n\}$, we have

$$x(\omega) = \sum_{n=1}^{N} c_n \psi^{(n)}(\omega \pm \mathrm{i}\epsilon) \sim \sum_{n=1}^{N} c_n \mathrm{e}^{-\mathrm{i}\omega s_n}(\omega \pm \mathrm{i}\epsilon)^{-\lambda_n} \mathrm{e}^{(n)}. \tag{40}$$

The addition of $\pm\mathrm{i}\epsilon$ (where ϵ is an arbitrarily small positive quantity) to ω in this formula is intended to indicate how the analytic continuation from positive to negative values of ω negotiates the singularity at $\omega = 0$ in each of the terms in the sums in (40). Because the sum as a whole has no singularity at $\omega = 0$ the choice of $+$ or $-$ is arbitrary so long as it is made consistently for all the terms in the sum. We can now show why the probability distribution $p(y)$ vanishes identically outside a certain range as predicted by our preliminary discussion in the previous section. We have

$$p(y) = \sum_{n=0}^{N} c_n \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega y} \psi^{(n)}(\omega \pm \mathrm{i}\epsilon). \tag{41}$$

If we make the negative choice then a typical term in the integrand will asymptotically have the form

$$c_n \mathrm{e}^{\mathrm{i}\omega(y-s_n)}(\omega - \mathrm{i}\epsilon)^{-\lambda_n}. \tag{42}$$

The ω-integration contour lies below the singularity at $\omega = 0$. When $y < s_n$ this contour can be closed in the lower half ω-plane thus encircling no singularities. From Cauchy's theorem we can conclude that the contribution to $p(y)$ from this term in the integral is zero. When $y > s_n$ the exponential factor does not permit the contour to be closed in the lower half ω-plane. Closure in the upper half ω-plane results in an enfolding by the integration contour of the singularity at $\omega = 0$ together with its attached branch-cut giving rise to a non-zero contribution to $p(y)$ from this term for $y > s_n$. We have then the following picture: assuming the choice of $-\mathrm{i}\epsilon$, for $y < \min\{s_n\}$ none of the terms in the sum in (41) yields a non-vanishing contribution. As we increase y through the range $\min\{s_n\} < y < \max\{s_n\}$ the terms are 'switched on' one by one. At this point it would appear that when $y > \max\{s_n\}$ all the terms contribute. In a sense this is the case. However we can show that in fact they sum up to zero. We do this by recalling that the choice between $+$ and $-$ is completely arbitrary. If we switch to the $+$ choice we are not changing the value of the integral but we are able, when $y > \max\{s_n\}$ to close the integration contour in the upper half ω-plane and, because of the absence of encircled singularities, to use the Cauchy argument to conclude that the resulting integral is again zero. It follows that our Fourier transform argument allows us to confirm what we argued on intuitive grounds in the previous section, that when all $\{\alpha_n\}$ are strictly positive the support for $p(y)$ lies in the range $\min\{s_n\} < y < \max\{s_n\}$ and that $p(y)$ vanishes identically outside this range.

When one of the $\{\alpha_n\}$ becomes negative two things happen simultaneously:

(i) one of the $\{\sigma_n\}$ becomes positive, the corresponding solution then vanishes at $\omega = 0$; and

(ii) one of the $\{\lambda_n\}$ becomes negative and the corresponding solution blows up as $\omega \to \infty$. (We will maintain the condition $Y'(0) > 0$ to avoid situations in which σ_n returns to the negative axis.) The boundary condition at $\omega = 0$ no longer picks out a unique solution since the requirement that $x(0) = q$ is satisfied by

$$x(\omega) = \phi^{(1)}(\omega) + \mu\phi^{(2)}(\omega) \tag{43}$$

for any value of the coefficient μ. However we can determine μ by adjusting its value so that $x(\omega)$ no longer has any contribution from the solution that blows at infinity. If two of the $\{\alpha_n\}$ become negative then there are two linearly independent solutions that vanish at $\omega = 0$. Now there are two parameters in the expression for $x(\omega)$. However now two of the $\{\lambda_n\}$ have become negative so there are two linearly independent solutions which must be excluded from contributing to $x(\omega)$. The two above parameters can be chosen to bring about this condition. Similar reasoning can be applied for any number M of negative $\{\alpha_n\}$ provided $0 \leqslant M \leqslant N-1$. If all of the $\{\alpha_n\}$ are negative then it is impossible to construct a physically acceptable solution for $x(\omega)$. Although the parameter count works for M in the above range it is not strictly possible at the present level of abstraction to prove that there really is a physically acceptable solution for these values of M. However if we make the assumption that there is a static probability distribution of the kind we seek then the proceedure we have described must find it.

Because we chose to specify the solutions $\Psi(\omega)$ by means of their asymptotic behaviour to the right, the above analysis only establishes the physical solution on the positive real ω-axis. The nature of the solution on the negative real ω-axis is determined by enforcing the requirement

$$x(-\omega) = [x(\omega)]^* \tag{44}$$

which is a necessary and sufficient condidition for the distribution $p(y)$ to be real.

Whatever the detailed structure of the physical solution at the origin it is clear that the non-analyticity introduced by the additional contribution to $x(\omega)$ affects the asymptotic behaviour of $p(y)$ for large y. From standard analysis of Fourier integrals a contribution with a singularity at $\omega = 0$ that behaves as $(\omega)^\sigma$ will produce asymptotic behaviour of the form

$$p(y) \sim \mathrm{O}(|y|^{-\sigma-1}) \qquad \text{as } y \to \infty. \tag{45}$$

Clearly the dominant asymptotic term will come from the contribution to $x(\omega)$ with the smallest (positive) value of σ_n. This change in the nature of the support for $p(y)$ from finite to infinite range together with the associated power-law fall-off as $|y| \to \infty$ is the major qualitative feature of these multiplicative stochastic differential equations. It represents a situation in which the distribution changes from having all its moments finite to one in which all the moments beyond a certain degree do not exist. It may be the case that the variance or even the mean of the distribution does not exist.

Fourier transform analysis also tells us that a contribution to $x(\omega)$ with an asymptotic form given in (42) will yield a term in $p(y)$ which behaves as

$$p(y) \sim |y - s_n|^{\lambda_n - 1} \qquad \text{as } y \sim s_n. \tag{46}$$

This indicates the nature of the non-analyticity of $p(y)$ at the points $y = s_n$ where the various contributions to the probability distribution are switched on and off. The implication for the first and last contributions which control behaviour at the ends of the range of support for $p(y)$ is particularly clear.

We will now illustrate these results by means of simple examples together with appropriate numerical simulations.

5. Specific examples

5.1. Two-component model

The very simplest case, which can treated completely explicitly, is the two-component case with $N = 2$. Except for special cases a shift in origin of y simply adds a multiple of the unit matrix to the matrix B. We will take advantage of this to set one of the elements of B to zero since this simplifies the algebra without losing anything essential. We therefore choose

$$A = \begin{pmatrix} A & 0 \\ 0 & \rho \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \qquad q = \begin{pmatrix} r \\ s \end{pmatrix} \tag{47}$$

where $r + s = 1$. According to the general theory set out in the previous sections we expect that when both $\alpha > 0$ and $\rho > 0$ the support of $p(y)$ will lie in the range $0 < y < \beta/\rho$ and when $\rho < 0$ the range will be extended to $0 < y < \infty$. From its definition in (27) we find in this case that

$$Y(\sigma) = 1 - \frac{r}{1 + \sigma\tau\alpha} - \frac{s}{1 + \sigma\tau\rho}. \tag{48}$$

The roots of $Y(\sigma)$ are $\sigma = \sigma_1 = 0$ and $\sigma = \sigma_2 = -(r/\tau\rho + s/\tau\alpha)$. Clearly as ρ passes through zero the second root changes sign from negative to positive and remains positive so long as $|\rho| < r\alpha/s$. Provided that this condition is respected we can apply this general theory to predict that when the probability distribution changes form it acquires for large positive y a power-law dependence of the form $p(y) \sim y^{-1-\sigma_2}$. These predictions will now be confirmed explicitly.

If we set

$$x(\omega) = \begin{pmatrix} u(\omega) \\ v(\omega) \end{pmatrix} \qquad \text{and} \qquad x^{(n)} = \begin{pmatrix} u_n \\ v_n \end{pmatrix} \tag{49}$$

then the recurrence relation (29) for the case $\sigma = 0$ implies

$$u_{n+1} = \frac{r}{\tau\alpha}\left(-\frac{\beta}{\rho}\right)\frac{1}{(n+1)(n+1+(r/\tau\rho)+(s/\tau\alpha))}v_n \tag{50}$$

and

$$v_{n+1} = \left(-\frac{\beta}{\rho}\right) \frac{(n+1+(s/\tau\alpha))}{(n+1)(n+1+(r/\tau\rho)+(s/\tau\alpha))} v_n . \tag{51}$$

The solution of these recurrence relations leads to the result

$$u(\omega) = r\frac{\Gamma(1+(r/\tau\rho)+(s/\tau\alpha))}{\Gamma(s/\tau\alpha)} \times \sum_{n=0}^{\infty} \left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^n \frac{\Gamma(n+(s/\tau\alpha))}{n!\Gamma(n+1+(r/\tau\rho)+(s/\tau\alpha))} \tag{52}$$

$$v(\omega) = s\frac{\Gamma(1+(r/\tau\rho)+(s/\tau\alpha))}{\Gamma(1+(s/\tau\alpha))} \times \sum_{n=0}^{\infty} \left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^n \frac{\Gamma(n+1+(s/\tau\alpha))}{n!\Gamma(n+1+(r/\tau\rho)+(s/\tau\alpha))}. \tag{53}$$

These solutions for u and v are confluent hypergeometric functions. In standard notation we have

$$\begin{pmatrix} u(\omega) \\ v(\omega) \end{pmatrix} = \begin{pmatrix} r\ {}_1F_1((s/\tau\alpha),(s/\tau\alpha)+(r/\tau\rho)+1,-(\beta/\rho)\mathrm{i}\omega) \\ s\ {}_1F_1((s/\tau\alpha)+1,(s/\tau\alpha)+(r/\tau\rho)+1,-(\beta/\rho)\mathrm{i}\omega) \end{pmatrix}. \tag{54}$$

It is easy to verify that the other linearly independent solution specified by its behaviour at $\omega = 0$ is, up to a normalization,

$$\begin{pmatrix} u(\omega) \\ v(\omega) \end{pmatrix} = (\mathrm{i}\omega)^{\sigma} \begin{pmatrix} (r/\tau\alpha)_1F_1(-(s/\tau\rho),1-(s/\tau\alpha)-(r/\tau\rho),-(\beta/\rho)\mathrm{i}\omega) \\ {}_1F_1(1-(r/\tau\rho),1-(s/\tau\alpha)-(r/\tau\rho),-(\beta/\rho)\mathrm{i}\omega) \end{pmatrix} \tag{55}$$

where $\sigma = \sigma_2 = -((s/\tau\alpha)+(r/\tau\rho))$. The asymptotic behaviour as $\omega \to \infty$ of the solution in (54) is to leading order

$$\phi^{(1)}(\omega) \sim \begin{pmatrix} (r\Gamma(s/\tau\alpha)+(r/\tau\rho)+1)/\Gamma((r/\tau\rho)+1) \\ 0 \end{pmatrix} \left(\frac{\beta}{\rho}\mathrm{i}\omega\right)^{-s/\tau\alpha} + \begin{pmatrix} 0 \\ (s\Gamma(s/\tau\alpha)+(r/\tau\rho)+1)/\Gamma((s/\tau\alpha)+1) \end{pmatrix} \times \left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^{-r/\tau\rho} \mathrm{e}^{-(\beta/\rho)\mathrm{i}\omega}. \tag{56}$$

The second solution in (55) has the asymptotic form

$$\phi^{(2)}(\omega) \sim \begin{pmatrix} ((r/r\alpha)\Gamma(1-(s/\tau\alpha)-(r/\tau\rho))/\Gamma(1-(s/\tau\alpha))) \\ 0 \end{pmatrix} \left(\frac{\beta}{\rho}\mathrm{i}\omega\right)^{-s/\tau\alpha} + \begin{pmatrix} 0 \\ (\Gamma((1-s/\tau\alpha)-(r/\tau\rho))/\Gamma(1-r/\tau\rho)) \end{pmatrix} \times \left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^{-r/\tau\rho} \mathrm{e}^{-(\beta/\rho)\mathrm{i}\omega}. \tag{57}$$

From (56) it is clear that when both $\alpha > 0$ and $\rho > 0$ the solution $\phi^{(1)}(\omega)$ is sufficiently bounded at large ω that it can be interpreted as the Fourier transform of a probability distribution. Using the standard representation of the confluent hypergeometric function we see that

$$\phi^{(1)}(\omega) = \frac{\Gamma((s/\tau\alpha)+(r/\tau\rho)+1)}{\Gamma(s/\tau\alpha)\Gamma(r/\tau\rho)} \int_0^1 \mathrm{d}t\, \mathrm{e}^{-(\beta/\rho)\mathrm{i}\omega t} t^{(s/\tau\alpha)-1}(1-t)^{(r/\tau\rho)-1} \times \begin{pmatrix} \tau\rho(1-t) \\ \tau\alpha t \end{pmatrix}. \tag{58}$$

With an appropriate scaling of the integration variable t this becomes immediately a Fourier expression with the result

$$p(y) = \frac{\Gamma((s/\tau\alpha)+(r/\tau\rho)+1)}{\Gamma(s/\tau\alpha)\Gamma(r/\tau\rho)} \left(\frac{\beta}{\rho}\right)^{-(s/\tau\alpha)-(r/\tau\rho)} \times \begin{pmatrix} \tau\rho y^{(s/\tau\alpha)-1}((\beta/\rho)-y)^{(r/\tau\rho)} \\ \tau\alpha y^{(s/\tau\alpha)}((\beta/\rho)-y)^{(r/\tau\rho)-1} \end{pmatrix} \tag{59}$$

for $0 < y < \beta/\rho$ and $p(y) = 0$ for y outside this range. It is also clear from (56) that $\phi^{(1)}(\omega)$ ceases to be Fourier transformable when ρ becomes negative. However as predicted from the general theory σ_2 becomes positive at this point and from (57) we see that we can construct a Fourier transformable function with the correct normalization by forming the combination

$$x(\omega) = \phi^{(1)}(\omega) - \frac{s\Gamma((s/\tau\alpha)+(r/\tau\rho)+1)\Gamma(1-(r/\tau\rho))}{\Gamma((s/\tau\alpha)+1)\Gamma(1-(s/\tau\alpha)-(r/\tau\rho))}\phi^{(2)}(\omega). \tag{60}$$

When $-\alpha < \rho < 0$ this solution has an integral representation of the form

$$x(\omega) = r\frac{\Gamma(r/\tau|\rho|)}{\Gamma(s/\tau\alpha)\Gamma((r/\tau|\rho|)-(s/\tau\alpha))} \times \begin{pmatrix} \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-(\beta/|\rho|)\mathrm{i}\omega u} u^{(s/\tau\alpha)-1}(1+u)^{-(r/\tau|\rho|)} \\ (\alpha/|\rho|)\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-(\beta/|\rho|)\mathrm{i}\omega u} u^{(s/\tau\alpha)}(1+u)^{-(r/\tau|\rho|)-1} \end{pmatrix}. \tag{61}$$

Again after appropriate scaling of the integration variable u we see that this is a Fourier transform formula which implies that

$$p(y) = r\frac{\Gamma(r/\tau|\rho|)}{\Gamma(s/\tau\alpha)\Gamma((r/\tau|\rho|)-(s/\tau\alpha))} \left(\frac{\beta}{|\rho|}\right)^{(r/\tau|\rho|)-(s/\tau\alpha)} \times \begin{pmatrix} y^{(s/\tau\alpha)-1}((\beta/|\rho|)+y)^{-(r/\tau|\rho|)} \\ (\alpha/|\rho|)y^{(s/\tau\alpha)}((\beta/|\rho|)+y)^{-(r/\tau|\rho|)-1} \end{pmatrix} \tag{62}$$

and $p(y) = 0$ for $y < 0$.

5.2. Three-component model

It is not in general easy to obtain explicit results for multi-component models. However there is a fairly general three component model which can be treated in a complete fashion and which reveals something of the complexity of the more elaborate models. We choose

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\beta \end{pmatrix} \qquad q = \begin{pmatrix} r \\ s \\ s \end{pmatrix} \tag{63}$$

where $r + 2s = 1$.

We find for the determinant function $Y(\sigma)$

$$Y(\sigma) = 1 - \frac{r}{1+\sigma\tau\alpha} - \frac{2s}{1+\sigma\tau\rho}. \tag{64}$$

The two solutions of this equation are $\sigma = \sigma_1 = 0$ and $\sigma = \sigma_2 = -(r/\tau\rho + 2s/\tau\alpha)$. The remaining solution of the indicial equation is the common value of the coincident poles of $Y(\sigma)$ namely $\sigma = \sigma_3 = -(1/\tau\rho)$. For the case in which both $\alpha > 0$ and $\rho > 0$, the Fourier transform of the probability distribution has the series expansion

$$x(\omega) = \begin{pmatrix} u(\omega) \\ v(\omega) \\ w(\omega) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} (\mathrm{i}\omega)^n. \tag{65}$$

The recurrence relation equation (29) implies for this case

$$u_{n+1} = -\frac{\tau\beta}{(1+(n+1)\tau\rho)} \frac{r(v_n - w_n)}{Y(n+1)(1+(n+1)\tau\alpha)} \tag{66}$$

$$v_{n+1} = -\frac{\tau\beta}{1+(n+1)\tau\rho}\left(v_n + \frac{s(v_n - w_n)}{Y(n+1)(1+(n+1)\tau\rho)}\right) \tag{67}$$

$$w_{n+1} = -\frac{\tau\beta}{1+(n+1)\tau\rho}\left(-w_n + \frac{s(v_n - w_n)}{Y(n+1)(1+(n+1)\tau\rho)}\right). \tag{68}$$

The latter two of these equations may be expressed in the convenient form

$$v_{n+1} - w_{n+1} = \left(-\frac{\beta}{\rho}\right) \frac{1}{n+1+(1/\tau\rho)}(v_n + w_n) \tag{69}$$

and

$$v_{n+1} + w_{n+1} = \left(-\frac{\beta}{\rho}\right) \frac{(n+1+(2s/\tau\alpha))}{(n+1)(n+1+(r/\tau\rho)+(2s/\tau\alpha))}(v_n - w_n). \tag{70}$$

In this form it is easy to solve the recurrence relations to yield the result

$$u(\omega) = \frac{1}{\sqrt{\pi}} \frac{rs}{\tau\alpha} \frac{\Gamma(1+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(1/2+1/2\tau\rho)}{\Gamma(1+(s/\tau\alpha))} \sum_{n=\mathrm{even}} \left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^n \times$$

$$\times \frac{1}{n!}\frac{\Gamma(n/2+(s/\tau\alpha))\Gamma(n/2+1/2)}{\Gamma(n/2+1+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(n/2+1/2+(1/2\tau\rho))} \tag{71}$$

$$v(\omega)+w(\omega)=\frac{1}{\sqrt{\pi}}\frac{rs}{\tau\alpha}\frac{\Gamma(1+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(1/2+1/2\tau\rho)}{\Gamma(1+(s/\tau\alpha))}$$
$$\times\sum_{n=\text{even}}\left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^n$$
$$\times \frac{1}{n!}\frac{\Gamma(n/2+1+(s/\tau\alpha))\Gamma(n/2+1/2)}{\Gamma(n/2+1+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(n/2+1/2+(1/2\tau\rho))} \tag{72}$$

and

$$v(\omega)-w(\omega)=\frac{1}{\sqrt{\pi}}\frac{rs}{\tau\alpha}\frac{\Gamma(1+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(1/2+1/2\tau\rho)}{\Gamma(1+(s/\tau\alpha))}$$
$$\times\sum_{n=\text{even}}\left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^n$$
$$\times \frac{1}{n!}\frac{\Gamma(n/2+1/2+(s/\tau\alpha))\Gamma(n/2+1)}{\Gamma(n/2+1/2+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(n/2+1+(1/2\tau\rho))}. \tag{73}$$

By making use of identities of the form

$$\frac{\Gamma(n/2+(s/\tau\alpha))}{\Gamma(n/2+1+(r/2\tau\rho)+(s/\tau\alpha))} = \frac{2}{\Gamma(1+(r/2\tau\rho))}\int_0^1 \mathrm{d}t\, t^{n+(2s/\tau\alpha)-1}(1-t^2)^{(r/2\tau\rho)} \tag{74}$$

and

$$\frac{\Gamma(n/2+1/2)}{\Gamma(n/2+1/2+(1/2\tau\rho))}=\frac{2}{\Gamma(1/2\tau\rho)}\int_0^1 \mathrm{d}u\, u^n(1-u^2)^{(1/2\tau\rho)} \tag{75}$$

we can derive integral representations for the solutions

$$u(\omega)=C\left(\frac{\rho}{\alpha}\right)\int_0^1\mathrm{d}t\int_0^1\mathrm{d}u\left(\mathrm{e}^{(-\beta/\rho)\mathrm{i}\omega tu}+\mathrm{e}^{(\beta/\rho)\mathrm{i}\omega tu}\right)$$
$$\times\, t^{(2s/\tau\alpha)-1}(1-t^2)^{(r/2\tau\rho)}(1-u^2)^{(1/2\tau\rho)-1} \tag{76}$$

$$v(\omega)+w(\omega)=C\int_0^1\mathrm{d}t\int_0^1\mathrm{d}u\left(\mathrm{e}^{-(\beta/\rho)\mathrm{i}\omega tu}+\mathrm{e}^{(\beta/\rho)\mathrm{i}\omega tu}\right)$$
$$\times\, t^{(2s/\tau\alpha)-1}(1-t^2)^{(r/2\tau\rho)}(1-u^2)^{(1/2\tau\rho)-1} \tag{77}$$

and

$$v(\omega) - w(\omega) = C\left(\frac{\rho}{\alpha}\right)\int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}u \left(\mathrm{e}^{(-\beta/\rho)\mathrm{i}\omega t u} + \mathrm{e}^{(\beta/\rho)\mathrm{i}\omega t u}\right) \times t^{(2s/\tau\alpha)-1}(1-t^2)^{(r/2\tau\rho)}(1-u^2)^{(1/2\tau\rho)-1} \tag{78}$$

where

$$C = \frac{4s}{\sqrt{\pi}} \frac{\Gamma(1+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(1/2+(1/2\tau\rho))}{\Gamma(1+(s/\tau\alpha))\Gamma((r/2\tau\rho)\Gamma(1/2\tau\rho))}. \tag{79}$$

If we introduce the function

$$\mathcal{F}_n(a,b,c,z) = \int_0^1 \mathrm{d}t \int_{-1}^1 \mathrm{d}u\, \mathrm{e}^{-ztu} t^{a-1}(1-t^2)^{b-1} u^{n-1}(1-u^2)^{c-1} \tag{80}$$

then we can express these solutions as

$$u(\omega) = C\frac{\rho}{\alpha}\mathcal{F}_1\left(\frac{2s}{\tau\alpha}+1, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, \frac{\beta}{\rho}\mathrm{i}\omega\right) \tag{81}$$

$$v(\omega) + w(\omega) = C\mathcal{F}_1\left(\frac{2s}{\tau\alpha}+2, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, \frac{\beta}{\rho}\mathrm{i}\omega\right) \tag{82}$$

$$v(\omega) - w(\omega) = C\mathcal{F}_2\left(\frac{2s}{\tau\alpha}+1, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, \frac{\beta}{\rho}\mathrm{i}\omega\right). \tag{83}$$

The integral representation for $\mathcal{F}$ in (80) allows us to compute its asymptotic behaviour for large imaginary argument. There are contributions to the asymptotic behaviour from certain sub-regions near the boundary of the integration region. These sub-regions are the two corner regions $(t,u) \sim (1-\mathrm{O}(\Omega^{-1}), 1-\mathrm{O}(\Omega^{-1}))$ and $(t,u) \sim (1-\mathrm{O}(\Omega^{-1}), -1+\mathrm{O}(\Omega^{-1}))$, together with the boundary strip $(t \sim \mathrm{O}(\Omega^{-1}), -1 < u < 1)$. The result is

$$\mathcal{F}_n(a,b,c,\mathrm{i}\Omega) \sim \frac{1}{4}\Gamma(b)\Gamma(c)\left(\frac{1}{(-\mathrm{i}\Omega)^{b+c}}\mathrm{e}^{\mathrm{i}\Omega} - (-1)^n \frac{1}{(\mathrm{i}\Omega)^{b+c}}\mathrm{e}^{-\mathrm{i}\Omega}\right) + \frac{\Gamma(a)\Gamma(c)\Gamma(n/2-a/2)}{\Gamma(n/2-a/2+c)}\left(\frac{1}{(\mathrm{i}\Omega)^a} - (-1)^n \frac{1}{(-\mathrm{i}\Omega)^a}\right). \tag{84}$$

These results imply contributions to the leading behaviour for $\omega \to \infty$ as follows

$$x(\omega) \sim C^{(1)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \left(\left(\frac{\beta}{\rho}\mathrm{i}\omega\right)^{-(2s/\tau\alpha)} + \left(-\frac{\beta}{\rho}\mathrm{i}\omega\right)^{-(2s/\tau\alpha)}\right) \tag{85}$$

$$x(\omega) \sim C^{(2)} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left(\mathrm{e}^{(\beta/\rho)\mathrm{i}\omega}\left(-\frac{\beta}{2\rho}\mathrm{i}\omega\right)^{-(1-s/\tau\rho)}\right) \tag{86}$$

$$x(\omega) \sim C^{(2)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(\mathrm{e}^{(\beta/\rho)\mathrm{i}\omega}\left(-\frac{\beta}{2\rho}\mathrm{i}\omega\right)^{-(1-s/\tau\rho)}\right) \tag{87}$$

where

$$C^{(1)} = \frac{4\tau\rho}{\sqrt{\pi}} \frac{\Gamma(1+(r/2\tau\rho)+(s/\tau\alpha))\Gamma(1/2+(1/2\tau\rho))\Gamma(1/2-(s/\tau\alpha))}{\Gamma(r/2\tau\rho)\Gamma(1/2-(s/\tau\alpha)+(1/\tau\rho))}. \quad (88)$$

and

$$C^{(2)} = \frac{s}{\sqrt{\pi}\alpha} \frac{\Gamma(1+(r/2\tau\rho)+1/2+(1/2\tau\rho))}{\Gamma(1+(s/\tau\alpha))}. \quad (89)$$

According to the general theory we can infer that our probability distribution vanishes outside the range $-\beta/\rho < y < \beta/\rho$ and at the end points behaves as $p(y) \sim |y \pm \beta/\rho|^{(1-s/\tau\rho)}$ while in the neighbourhood of $y = 0$ we have $p(y) \sim |y|^{(1-r/\tau\alpha)}$. We can easily find a representation for $p(y)$ by Fourier transforming $x(\omega)$. It is convenient to define

$$\mathcal{G}_n(a,,b,c,y) = \int \frac{d\omega}{2\pi} e^{i\omega y} \mathcal{F}_n\left(a,b,c,i\frac{\beta}{\rho}\omega\right) \quad (90)$$

which yields

$$\mathcal{G}_n(a,b,c,y) = \int_0^1 dt \int_{-1}^1 \delta\left(y - \frac{\beta}{\rho}tu\right) t^{a-1}(1-t^2)^{b-1}u^{n-1}(1-u^2)^{c-1}. \quad (91)$$

After eliminating the delta-function by performing the u-integration and making the following change of integration variable

$$t \to t' = \left(t^2 - \left(\frac{\rho|y|}{\beta}\right)^2\right)\left(1 - \left(\frac{\rho|y|}{\beta}\right)^2\right)^{-1} \quad (92)$$

we find

$$\begin{aligned} \mathcal{G}_n(a,b,c,y) = {} & \frac{\Gamma(c)\Gamma(c+b-(a-n/2))}{\Gamma(2c+b-(a-n/2))} \\ & \times \frac{\rho}{2\beta}\left(\frac{\rho|y|}{\beta}\right)^{a-2c-1}\left(1-\left(\frac{\rho|y|}{\beta}\right)^2\right)^{b+c-1} \\ & \times F\left(c-\frac{a-n}{2}, c, c+b-\frac{a-n}{2}, 1-\left(\frac{\beta}{\rho|y|}\right)^2\right) \end{aligned} \quad (93)$$

where F is the standard hypergeometric function. The behaviour of $\mathcal{G}_n$ at $y = \pm\beta/\rho$ is immediately evident from this equation. Its behaviour near $y = 0$ can be inferred from the general theory of the hypergeometric function. The relevant part of its behaviour is $\mathcal{G}_n \sim |y|^{a-1}$. Using equations (81)–(83) the probability distribution can be expressed as

$$p(y) = C\begin{pmatrix} \frac{\rho}{\alpha}\mathcal{G}_1\left(\frac{2s}{\tau\alpha}+1, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, y\right) \\ \frac{1}{2}\mathcal{G}_1\left(\frac{2s}{\tau\alpha}+2, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, y\right) + \frac{1}{2}\mathcal{G}_2\left(\frac{2s}{\tau\alpha}+1, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, y\right) \\ \frac{1}{2}\mathcal{G}_1\left(\frac{2s}{\tau\alpha}+2, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, y\right) - \frac{1}{2}\mathcal{G}_2\left(\frac{2s}{\tau\alpha}+1, \frac{r}{2\tau\rho}, \frac{1}{2\tau\rho}, y\right) \end{pmatrix}. \quad (94)$$

By making use of the information in these equations it is straightforward to verify the predicted behaviour of $p(y)$.

This analysis only holds when $\rho > 0$. When $\rho < 0$ the asymptotic behaviour of $x(\omega)$ for $\omega \to \infty$ becomes more singular and it ceases to be a Fourier transformable function. One sign of this change is that the integral representations for the components of $x(\omega)$ in (81)–(83) cease to be convergent as a result of divergences at the boundaries where $t = 1$ or $u = 1$. We can extract a representation of the new form of $x(\omega)$ from these old representations by detaching the t- and u-integration contours from the end points $t = 1$ and $u = 1$, rotating them in their respective complex planes $t \to e^{i(\pi/2)}t$ and $u \to e^{-i(\pi/2)}u$ and subsequently extending the contours to infinity. The resulting representations for the components of $x(\omega)$ are

$$u(\omega) = C'\left(\frac{|\rho|}{\alpha}\right)\int_0^\infty dt\int_0^\infty du\left(e^{-(\beta/|\rho|)i\omega tu} + e^{(\beta/|\rho|)i\omega tu}\right) \times t^{(2s/\tau\alpha)-1}(1+t^2)^{-(r/2\tau|\rho|)}(1+u^2)^{-(1/2\tau|\rho|)-1} \tag{95}$$

$$v(\omega) + w(\omega) = C\int_0^\infty dt\int_0^\infty du\left(e^{-(\beta/|\rho|)i\omega tu} + e^{(\beta/|\rho|)i\omega tu}\right) \times t^{(2s/\tau\alpha)-1}(1+t^2)^{-(r/2\tau|\rho|)}(1+u^2)^{-(1/2\tau|\rho|)-1} \tag{96}$$

and

$$v(\omega) - w(\omega) = C\int_0^\infty dt\int_0^\infty du\left(e^{-(\beta/|\rho|)i\omega tu} + e^{(\beta/|\rho|)i\omega tu}\right) \times t^{(2s/\tau\alpha)-1}(1+t^2)^{-(r/2\tau|\rho|)}(1+u^2)^{-(1/2\tau|\rho|)-1} \tag{97}$$

where

$$C = \frac{4s}{\sqrt{\pi}}\frac{\Gamma(1+(r/2\tau|\rho|)+(s/\tau\alpha))\Gamma(1/2+(1/2\tau|\rho|))}{\Gamma(1+(s/\tau\alpha))\Gamma(r/2\tau|\rho|)\Gamma(1/2\tau|\rho|)}. \tag{98}$$

They are convergent when $\rho < 0$ since the original boundary giving rise to the divergence has been eliminated and the integrands vanish as $t \to \infty$ and $u \to \infty$ at least for an appropriately restricted range for ρ. We will not examine the new probability distributions in detail but it is already clear that the support extends to infinity and has a power-law fall-off $p(y) \sim |y|^{(1-r/\tau\alpha)-(r/\tau|\rho|)}$.

6. Numerical simulations

Whatever the complexities of this analysis it is easy to simulate these simple models of multiplicative stochastic differential equations. We have plotted the distributions appropriate to the transformed variable θ where $y = \tan(\theta/2)$ in order to contain the whole graph in a finite range. The results for the two-component model are shown in figure 1 for the parameter set $\alpha = 1, \rho = 1, \beta = 1$ and $\tau = 0.5$. The plotted points represent the heights and centres of histogram blocks of width $\pi/20$. Statistical errors are smaller than the size of the plotted points. Note that as predicted, when $\rho > 0$ the distribution does indeed vanish beyond the point $\theta = \pi/4$ which for these parameters corresponds to $y = \beta/\rho$. The results are compared to the predictions

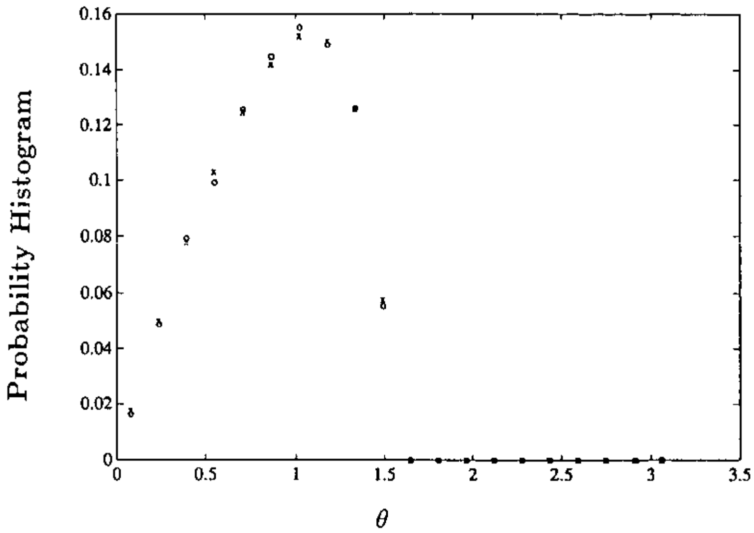

Figure 1. The probability distribution for the two-component model with parameters $\alpha = 1$, $\rho = 1$, $\beta = 1$ and $\tau = 0.25$. The simulation results are indicated by (o) and the exact results by (x).

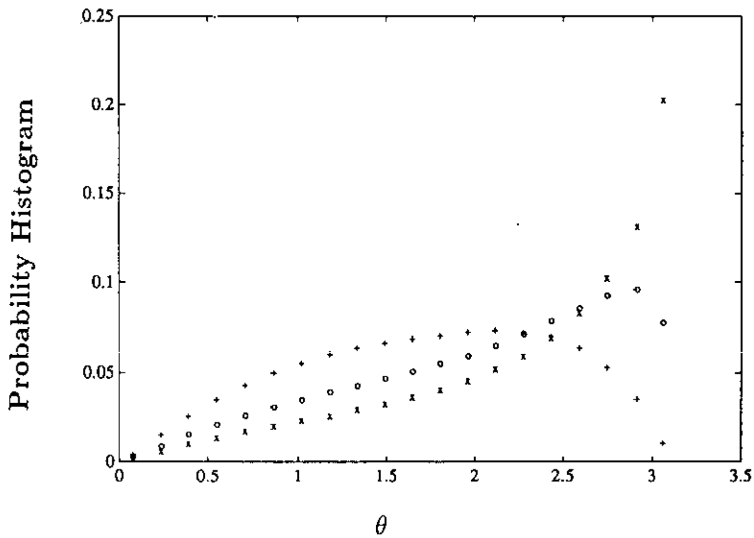

Figure 2. The simulation results for the probability distribution for the two-component model. The parameters are $\alpha = 2$, $\beta = 1$ and $\tau = 0.1$. The values for ρ are indicated by (+) for $\rho = -1$, (o) for $\rho = -1.3$, (x) for $\rho = -1.5$.

of the exact probability distribution for the histogram. The comparison is relatively good even though there is a small systematic error on some of the central points. Figure 2 exhibits the distribution for negative values of ρ. Note that as ρ is reduced the distribution is increasingly biased to $\theta = \pi$ which corresponds to $y \to \infty$.

It is clear from figures 3 and 4 that similar results are obtained for the three-component model where the passage from bounded to unbounded support for $p(y)$

is again observed as ρ passes from positive to negative values. Results from the simulation of models with larger numbers of components exhibit the same qualitative behaviour.

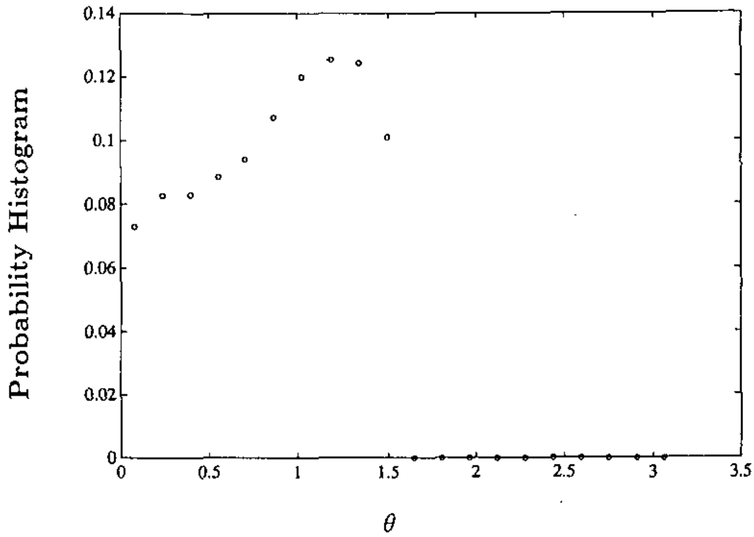

Figure 3. The simulation results for the probability distribution for the three component model with parameters $\alpha = 1$, $\rho = 3$, $\beta = 1$ and $\tau = 0.25$. Note that the distribution vanishes beyond the point $\theta = \pi/4$ in agreement with the theoretical prediction.

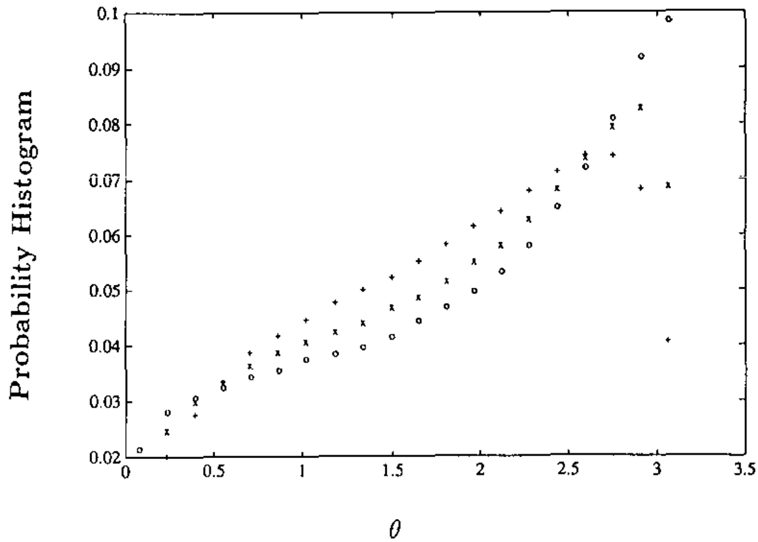

Figure 4. The simulation results for the probability distribution for the three component model with parameters $\alpha = 4$, $\rho = -1$ and $\beta = 1$. The values for ρ are indicated by (+) for $\tau = 0.1$, (x) for $\tau = 0.125$, (o) for $\tau = 0.15$.

7. Perturbation theory

In the limit in which $\tau \to 0$ we can expect the total probability distribution for y $P(y)$, to obey a diffusion equation. We can get some understanding of the nature of this limit by examining the two-component model. We have for the case $\alpha > 0$ and $\rho > 0$,

$$P(y) = u^{\mathrm{T}} p(y) \propto y^{(s/\tau\alpha)} \left(\frac{\beta}{\rho} - y\right)^{(r/\tau\rho)} \left(\frac{\rho}{y} + \frac{\alpha}{((\beta/\rho) - y)}\right). \quad (99)$$

To leading order in $1/\tau$

$$\log\left(P(y)\right) \sim \phi(y) \equiv \frac{s}{\tau\alpha}\log(y) + \frac{r}{\tau\rho}\log\left(\frac{\beta}{\rho} - y\right). \quad (100)$$

The stationary point of $\phi(y)$ occurs where the derivative

$$\phi'(y) = \frac{s}{\tau\alpha}\left(\frac{1}{y}\right) - \frac{r}{\tau\rho}\left(\frac{1}{((\beta/\rho) - y)}\right) = 0. \quad (101)$$

This occurs at the point $y = y_0$ where

$$y_0 = \frac{s\beta}{r\alpha + s\rho}. \quad (102)$$

We also have

$$\phi''(y_0) = -\frac{1}{\tau}\frac{(r\alpha + s\rho)^3}{rs\alpha^2\beta^2}. \quad (103)$$

The asymptotic shape of the distribution is a Gaussian

$$P(y) \sim \exp\left(-\frac{1}{2\tau}\frac{(r\alpha + s\rho)^3}{rs\alpha^2\beta^2}(y - y_0)^2\right). \quad (104)$$

In the limit of small τ the mean of the distribution is $\langle y \rangle = y_0$ and the variance $\sigma_y^2 \sim \mathrm{O}(\tau)$. This suggests we can derive a diffusion equation for $P(y)$ for values of $y = y_0 + \mathrm{O}(\sqrt{\tau})$. We can achieve this using a perturbation method by setting $y = y_0 + \sqrt{\tau} z$ in (10). The equation for $p(y)$ becomes

$$\frac{\partial}{\partial t}p(y) = \frac{\partial}{\partial z}\left(Az + \frac{1}{\sqrt{\tau}}(Ay_0 - B)\right)p(y) - \frac{1}{\tau}(\mathbf{1} - qu^{\mathrm{T}})p(y). \quad (105)$$

We now expand $p(y)$ as a power series

$$p(y) = \sum_{n=0}^{\infty} p_n(z)(\sqrt{\tau})^n. \quad (106)$$

On substituting this expansion into (105) and equating the coefficients of $(\sqrt{\tau})^n$ to zero we find

$$0 = -\left(\mathbf{1} - qu^{\mathrm{T}}\right) p_0(z) \tag{107}$$

$$0 = \frac{\partial}{\partial z}\left(Ay_0 - B\right) p_0(z) - \left(\mathbf{1} - qu^{\mathrm{T}}\right) p_1(z) \tag{108}$$

$$\frac{\partial}{\partial t} p_0(z) = \frac{\partial}{\partial z} Az p_0(z) + \frac{\partial}{\partial z}\left(Ay_0 - B\right) p_1(z) - \left(\mathbf{1} - qu^{\mathrm{T}}\right) p_2(z). \tag{109}$$

From (107) we see that $p_0(z) = qP_0(z)$. Multiplying (108) on the left by u^{T} and using this form for $p_0(z)$ we find

$$0 = \frac{\partial}{\partial z} P_0(z) u^{\mathrm{T}}\left(Ay_0 - B\right) q. \tag{110}$$

Since $\partial/\partial z\, P_0(z)$ does not vanish identically we must choose y_0 so that

$$u^{\mathrm{T}}\left(Ay_0 - B\right) q = 0. \tag{111}$$

That is $y_0 = u^{\mathrm{T}} Bq / u^{\mathrm{T}} Aq = \langle B\rangle / \langle A\rangle$. We can also conclude from (108) that if we set $p_1(z) = qP_1(z) + r_1(z)$ where $u^{\mathrm{T}} r_1(z) = 0$ then

$$r_1(z) = \frac{\partial}{\partial z}\left(Ay_0 - B\right) q P_0(z). \tag{112}$$

Multiplying (109) on the left by u^{T} we find

$$\frac{\partial}{\partial t} P_0(z) = \frac{\partial}{\partial z} u^{\mathrm{T}} Aq P_0(z) + \frac{\partial}{\partial z} u^{\mathrm{T}}(Ay_0 - B)(qP_1(z) + r_1(z)). \tag{113}$$

Using (112) we find

$$\frac{\partial}{\partial t} P_0(z) = \frac{\partial}{\partial z}\left[u^{\mathrm{T}} Aqz + u^{\mathrm{T}}(Ay_0 - B)^2 q \frac{\partial}{\partial z}\right] P_0(z). \tag{114}$$

Translating back into the original variables and using the approximation $P(y) = \sqrt{\tau} P_0(z)$ we find

$$\frac{\partial}{\partial t} P(y) = \frac{\partial}{\partial y}\left[\langle (Ay - B)\rangle + \tau \langle (Ay_0 - B)^2\rangle \frac{\partial}{\partial y}\right] P(y). \tag{115}$$

This is very close to the effective diffusion equation derived by van Kampen [1] using a similar perturbation method. However in our derivation we have tried to embody the idea that such an equation can only be valid in a neighbourhood of the point y_0. Were we to attempt a higher approximation we would obtain an equation for $P(y)$ involving higher derivatives than the second. The static solution of this equation is

$$P(y) \propto \exp\left\{-\frac{(\langle A\rangle y - \langle B\rangle)^2}{2\tau\langle A\rangle\langle (Ay_0 - B)^2\rangle}\right\}. \tag{116}$$

For the two-component model $\langle A\rangle = r\alpha + s\rho$ and $\langle B\rangle = s\beta$. This sets $y_0 = s\beta/(r\alpha + s\rho)$ which is indeed the centre of the limiting distribution we calculated previously. We also have $\langle (Ay_0 - B)^2\rangle = rs\alpha^2\beta^2/(r\alpha + s\rho)^2$ which leads to the variance appropriate to the form of the limiting distribution in (104). In this sense the perturbatively derived effective diffusion equation correctly describes the small τ-limit. The diffusion process naturally cannot be described when far away from the neighbourhood of the Gaussian peak. In this connection it is interesting to examine the limiting form of the stationary distribution when $\rho < 0$. We have from (62)

$$P(y) \propto y^{(s/\tau\alpha)} \left(\frac{\beta}{|\rho|} + y\right)^{(-r/\tau|\rho|)} \left(\frac{|\rho|}{y} + \frac{\alpha}{((\beta/|\rho|) + y)}\right). \tag{117}$$

In the limit $\tau \to 0$ the asymptotic shape of $P(y)$ is

$$P(y) \propto \exp\left\{-\frac{1}{2\tau}\frac{(r\alpha - s|\rho|)^3}{rs\alpha^2\beta^2}(y - y_0)^2\right\}. \tag{118}$$

This is really the same as (104) with $\rho \to -|\rho|$. It follows immediately that the right-hand side of (118) satisfies (115) the effective diffusion equation. However it should be noted that the application of the limit $\tau \to 0$ is non-uniform because of the nature of the original probability distribution in this case. We recall what is obvious from (117) namely, that because the distribution now has a power-law fall-off as $y \to \infty$, $\langle y^n\rangle$ will, for sufficiently large n, cease to exist. This remains true for any value of τ no matter how small. As $\tau \to 0$ the value of n for which $\langle y^n\rangle$ diverges, goes to ∞. Nevertheless there could be a situation in which τ was small enough for the Gaussian shape in (116) to be a good approximation for y near y_0 but for $\langle y^n\rangle$ to diverge for some relevant value of n.

The limit of $\tau \to \infty$ is also of interest and we will give a brief discussion of it. When $\tau \to \infty$ (22) becomes

$$\mathrm{i}\omega\left(-\frac{\partial}{\partial \mathrm{i}\omega}A - B\right)x(\omega) = 0. \tag{119}$$

Because of the boundary condition $x(0) = q$, the correct solution is

$$x(\omega) = \exp\left\{-A^{-1}B\mathrm{i}\omega\right\}q. \tag{120}$$

This leads to a total probability distribution

$$P(y) = \sum_{n=1}^{N} q_n\delta\left(y - \frac{\beta_n}{\alpha_n}\right). \tag{121}$$

We can see how this comes about in the two component model for the case $\alpha > 0$ and $\rho > 0$. Taking account of the relevant factors in the normalization constant in (59) we see that for large τ

$$P(y) \propto \left(\frac{1}{\tau\alpha}y^{(s/\tau\alpha)-1}\left(\frac{\beta}{\rho} - y\right)^{(r/\tau\rho)} + \frac{1}{\tau\rho}y^{(s/\tau\alpha)}\left(\frac{\beta}{\rho} - y\right)^{(r/\tau\alpha)-1}\right). \tag{122}$$

For large τ this distribution is $O(1/\tau)$ except for the strong peaks at $y = 0$ and $y = \beta/\rho$. In the limit $\tau \to \infty$ the peaks do become δ-functions and the probability distribution does assume the predicted form

$$P(y) = r\delta(y) + s\delta\left(\frac{\beta}{\rho} - y\right). \tag{123}$$

This mechanism is similar to one discussed in a previous paper [8]. When similar reasoning is applied to the case where $\rho < 0$ we see from (117) that the only peak is at $y = 0$ and the probability distribution acquires the form

$$P(y) = r\delta(y). \tag{124}$$

The interpretation of this result is that, although $P(y)$ away from $y = 0$ is $O(1/\tau)$, *the total weight in the power-law tail is* s. This fraction of the ensemble is driven off to infinity as $\tau \to \infty$. Another way of saying this is that the predicted s-function at $y = -\beta/|\rho|$ is unstable in the sense that the y-velocity field diverges from rather converges on this point.

8. Conclusion

We have investigated a class of linear multiplicative stochastic differential equations. An important property of these equations is the occurrence of a transition in the nature of the probability distribution for the dependent variable as some of the parameters of the controlling stochastic process are changed. The effect resembles a noise-induced transition but is dependent on the range of the relevant variables in the stochastic process rather than their means and variances. The transition is of a dramatic character in that the probability distribution changes from one with all its moments finite to one with at most a finite number of convergent moments as the result of the appearance of an inverse power-law tail for large values of the dependent variable. We illustrated these results with appropriate numerical simulations.

The tractability of the analysis of the stochastic differential equation depended on the simplification that the driving stochastic process took its values in a finite set, which we refer to as components. We were able to obtain explicit representations for the solutions for a two- and a three-component model. Success in the three-component case encourages us to believe that it should prove possible to obtain integral representations for the solutions of N-component models. This certainly is a problem worth pursuing since its solution would provide a basis for attacking those problems in which the driving stochastic process takes values that are denumerably infinite or continuously distributed. The basic phenomenon of transition from a confined distribution to a strongly extended distribution must occur more generally and it would be of great interest to obtain knowledge of such effects. One can visualize the power-law tail in the probability distribution being modified by logarithmic factors, for example.

In this paper we restricted our attention to equations where the dependent variable was a scalar quantity. Of even more interest are problems where this variable itself is a vector quantity. Transition phenomena of a similar kind must also occur in this vector case. One way of seeing this is to examine the equation for the length of the vector variable. We have for the (vector) y

$$\dot{y} = -A(t)y + B(t) \tag{125}$$

where now $A(t)$ is a matrix and $B(t)$ is a vector. The length of y is $\xi = \sqrt{(y^{\mathrm{T}}y)}$. If we set $y = \xi w$ where w is a unit vector in the direction of y then we find

$$\dot{\xi} = -(w^{\mathrm{T}}Aw)\xi + w^{\mathrm{T}}B. \tag{126}$$

This equation bears a resemblance to the original stochastic differential for a one-dimensional variable. Relying on this analogy we can conclude that if the length of B is bounded and if the eigenvalues of the symmetric part of A are positive and not too broadly distributed then the resulting distribution for ξ should be essentially bounded at least in the sense of having an entire set of finite moments. However if the symmetric part of A can acquire negative eigenvalues we expect $(u^{\mathrm{T}}Au)$ to fluctuate in sign and as a consequence, the resulting distribution for ξ to become extended and develop a power-law tail as $\xi \to \infty$. A phenomenon of this kind has been observed in the simulation of curvature evolution in material elements transported in random flows [3, 4, 6].

Finally we draw attention to the point that for parameter ranges of the driving stochastic process which result in a bounded distribution for the dependent variable the perturbation analysis in terms of powers of the correlation time works well in the neighbourhood of the Gaussian peak. For cases where the dependent variable acquires a broadly spread distribution the perturbation series works only in a non-uniform way and may require care in its application.

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